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A GEOMETRIC ILLUSTRATION OF LIMITS.

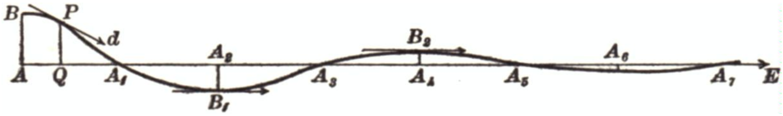
By Carl Eben Stromquist.

Teachers of freshman mathematics, the calculus, or review courses for high school teachers often experience some difficulty when the topic of limits comes up for discussion. The difficulty arises from the fact that the teacher must first "unteach" an erroneous definition of limits which is still to be found in many of our high school texts. The following definition, taken from a rather widely used plane geometry text, is typical of the error. "The limit of a variable is a constant from which the variable can be made to differ by less than any assigned quantity, but to which it can never be made equal." In a random selection of ten texts five gave essentially this definition. One of these was published as recently as 1915. The mischief is caused by the last part of the statement, viz., "but to which it can never be made equal."

Just why this part ever was or at present is included in the definition is difficult to explain, unless it is because in practically all the theorems and problems where the definition is needed and that the student meets in his high school mathematics the variable does not reach the limit. Thus, the perimeter, or area, of a regular polygon inscribed in a circle never reaches the circumference, or area, of the circle. As a matter of fact, geometric illustrations of cases where the variable does reach the limit are almost entirely lacking in our elementary texts. It is the purpose of this article to present an illustration which the writer has found helpful in his classes.

Before giving the illustration it may be well to state a definition of limit which is free from the objection already raised. The following will suffice for elementary purposes at least. If a variable x represents any one of an infinite series of values, it is said to approach a constant quantity a as a limit provided that the numerical value of the difference $x - a$ finally becomes and remains less than any preassigned positive quantity. It is important to observe the following three conditions which are involved in this definition, viz., that x must vary according to some law, that the difference $x - a$ must become numerically less than any assigned number, and, thirdly, that as x continues to vary the difference $x - a$ must remain less than this number.

To illustrate how a variable may satisfy the preceding definition of a limit and still reach the limit, imagine a race track constructed according to the following specifications. A line AE is drawn in an easterly direction from a given point A. On this line equal segments, say 100 yards, AA_1, A_1A_2, A_2A_3, etc., are laid off, and at A, A_2, A_4, etc., perpendiculars AB, A_2B_1, A_4B_2, etc., are laid off alternately to the north and to the south of the line AE, and such that each of these perpendiculars is equal to one half of the preceding one. The race track is now to be constructed by drawing a smooth curve through the points B, A_1, B_1, A_3, B_2, A_5, etc., as in the figure below. The direction d at which a horse will be running at any point P of the race track will be the direction of the tangent line drawn to the curve at this point. This direction will vary for different positions of P, but it is evident that, as P moves further and further away from B, the variable d will approach more and more an easterly direction, *i. e.*, a direction parallel to the line AE. It is evident, intuitively, that the definition of a limit is here satisfied. The variable d, however, reaches the limit at all the points B, B_1, B_2, etc.



Attention might also be called to the fact that, if a perpendicular PQ be drawn from P to AE, this perpendicular approaches zero as a limit as P moves away from B. Here again the variable PQ reaches its limit, zero, at the points A_1, A_3, A_5, etc.

The race track as here described depends upon a free hand construction. A more accurate construction may be obtained by drawing the graphs of certain equations which the student may have occasion to study later, such as, $y = \sin x/x$, or $y = e^{-x} \sin x$.

UNIVERSITY OF WYOMING,
LARAMIE, WYOMING.